Universal persistence exponents in an extremally driven system

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(Received 18 October 2001; published 25 January 2002)

The local persistence $R(t)$, defined as the proportion of the system still in its initial state at time t, is measured for the Bak-Sneppen model. For one and two dimensions, it is found that the decay of $R(t)$ depends on one of two classes of initial configuration. For a subcritical initial state, $R(t)\sim t^{-\theta}$, where the persistence exponent θ can be expressed in terms of a known universal exponent. Hence θ is universal. Conversely, starting from a supercritical state, $R(t)$ decays by the anomalous form $1-R(t)\sim t^{\tau_{\rm all}}$ until a finite time t_0, where $\tau_{\rm all}$ is also a known exponent. Finally, for the high dimensional model $R(t)$ decays exponentially with a nonuniversal decay constant.

DOI: 10.1103/PhysRevE.65.027104 PACS number(s): 05.70.Ln, 64.60.Ht

Extremally driven systems represent an important class of nonequilibrium models, with applications including sheared granular media, biological macroevolution and interface depinning [1–4]. Their defining characteristic is that they are driven in the vicinity of an *extremal* quantity, often the minimum or maximum of an inhomogeneous scalar field. Such deterministic dynamics can be naturally realized in the limit of vanishing (thermal) noise [5–7]. Moreover, many extremal models are also "critical" in an analogous sense to a continuous phase transition in equilibrium systems [8].

However, the temporal evolution of these systems is in many cases not fully understood. This is clearly demonstrated by recent work on one of the simplest, and therefore most studied, extremally driven systems, namely, the Bak-Sneppen model [3]. This has been shown to *age* in a similar manner to glassy systems [9,10], suggesting that it does not reach a steady state in the time frame of current simulation methods. This is in contrast to the usual claim that stationarity is reached "after an extensive transient" [3]. It now appears that this contradiction arises because most studies only measure functions of a single time variable, such as critical exponents, etc., that can appear to become constant after a finite time even for a nonstationary system; a true test of stationarity (such as the aging result mentioned above) requires the measurement of two-time correlation functions [11,12].

The aim of this paper is to investigate one aspect of the temporal behavior of the Bak-Sneppen model that has not yet been studied, in the hope that this will further elucidate the time evolution of extremally driven systems in general. The quantity that we consider is the local *persistence* $R(t)$, defined as the proportion of the system at a time t that has not yet changed from its initial state. This quantity has been the focus of considerable attention in recent years, at least in part because it is often found to decay algebraically with a nontrivial *persistence exponent* θ, $R(t)\sim \text{const}\times t^{-\theta}$, that (for equilibrium systems) cannot be related to either the static or dynamic exponent [13–16]. The universality of θ remains an open subject. In many cases it is not universal; that is, it may depend on the lattice coordination number or the precise choice of interaction term [16,17]. However, it has been found to be universal in directed percolation [16]. We find this also to be the case for the Bak-Sneppen model, and confirm our numerical finding by deriving an expression for θ in terms of the known universal exponent γ [2]. We further find that, when starting from a supercritical initial configuration, $R(t)$ decays in an anomalous, nonalgebraic manner. Finally, $R(t)$ always decays exponentially in the infinite dimensional case.

The Bak-Sneppen model is defined as follows [2,3]. A scalar quantity $f_i\in[0,1]$ is independently assigned to every site i of a d-dimensional hypercubic lattice, which contains a total of N sites. For every unit time t, the smallest f_i in the system is found, and it and its nearest neighbors are assigned new values, again drawn uniformly from $[0,1]$. Other initial conditions and interaction terms will be considered below. To generate the random values of the f_i, we used L'Ecuyer's 64-bit combined multiple recursive pseudo-random-number generator MRG63k3a, which has a quoted period of 2^{377} [18]. The typical number of operations required to find the global minimum was reduced from $\mathcal{O}(N)$ to $\mathcal{O}(\ln N)$ by using a binary search tree, which also constrains the system size to be a power of 2 [19].

Typical results for a periodic one-dimensional (1D) lattice are presented in Fig. 1. It can clearly be seen that the curves of $R(t)$ for different system sizes collapse after rescaling the time scale by N, for all $N>2^{10}\approx 10^3$. For the largest system $N=2^{21}$, the simulation time was long and only one run was performed; however, for smaller systems $R(t)$ was averaged over 10–100 different initial configurations. Also plotted in the figure is the theoretical prediction $R(t)\propto t^{-1.59}$, which will be derived below. The numerical $R(t)$ appear to be converging toward this prediction at late times, but there is still a slight curvature to the lines (on log-log axes) even for the largest t we were able to simulate, making direct measurement of the exponent difficult.

A similar picture emerges in two dimensions, as shown in Fig. 2. The theoretical prediction in this case is $R(t)\propto t^{-2.43}$ (again, see below for details), which is consistent with the simulation data, although as before there is still some slight curvature to $R(t)$ for the largest t available to our current computing resources.

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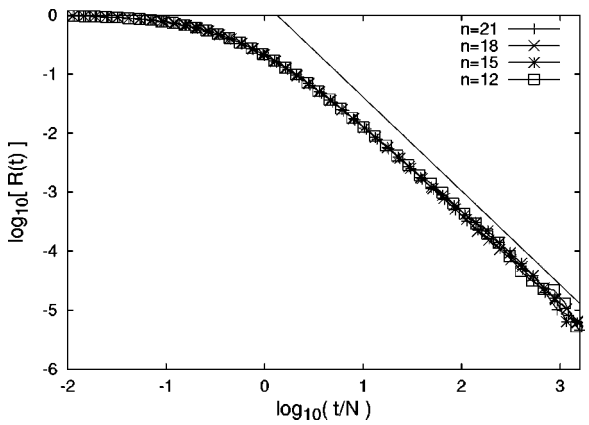

FIG. 1. The persistence $R(t)$ against the rescaled time variable t/N for different system sizes $N=2^n$, with $n=12$, 15, 18, and 21 as shown. The thin solid line has a slope of -1.59.

The universality of the exponent was tested by altering the microscopic details of the interactions. Three alternative rules were considered: updating the second nearest neighbors as well as the nearest neighbors (''2nd NN's''); changing the value of the nearest neighbors from f_i to $\frac{1}{2}(f_i+r)$, where the annealed random variable r is uniformly distributed on [0,1] (''Half NN''); and setting the minimum value to 1 but changing the nearest neighbors as normal (''Min to 1''). In all cases θ was found to be approximately the same. This has been tested in both one and two dimensions; the 1D case is shown in Fig. 3.

We shall now explain how the persistence exponent can be theoretically derived. This argument utilizes known results for the Bak-Sneppen model, which for reasons of space will not be repeated here; the reader is instead referred to [2,3,20] for their justification. It is known that the location of the minimum or ''active'' site initially jumps around the lattice in an uncorrelated manner, but as the system evolves the lattice variables f_i become spatially correlated, and the active site tends to remain in a localized region of the lattice for a finite amount of time before jumping to another, uncorrelated region of the lattice. Each such period of localized activity is known as an ''avalanche.''

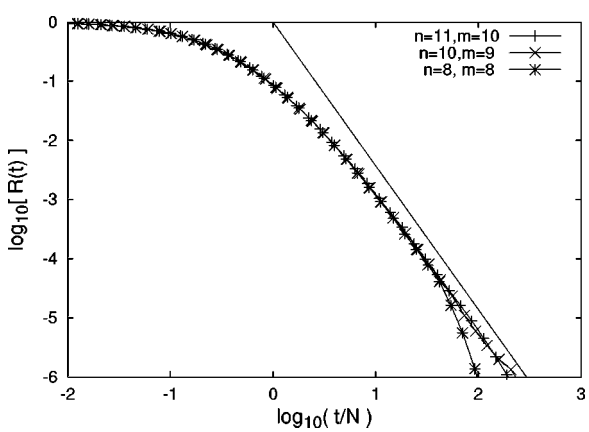

FIG. 2. The persistence $R(t)$ for two-dimensional lattices of sizes $N=2^m\times 2^n$. The solid straight line has a slope of -2.43. The data were averaged over at least ten separate runs.

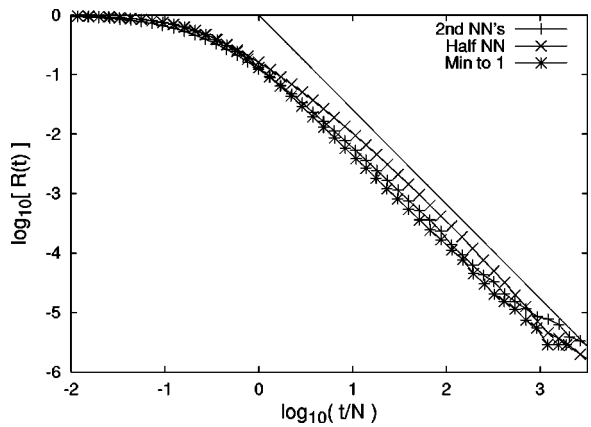

FIG. 3. $R(t)$ for one-dimensional systems of size $L=2^{21}$ with different interaction rules, to determine the universality of the persistence exponent. See the main text for explanation of the terms ''2nd NN,'' ''Half NN,'' and ''Min to 1.'' The solid line has a slope of -1.59.

Let the mean duration of an avalanche at time t be denoted by $\langle S(t)\rangle$, and the mean number of sites changed at least once during the avalanche by $\langle n_{\rm cov}(t)\rangle$, where the angular brackets refer to averages over different initial configurations. By taking a suitably large system size N, these quantities will increase by an arbitrarily small amount between avalanches, allowing the continuum limit to be taken. It is known that both quantities increase algebraically in time [2],

$$\langle S(t)\rangle \sim A\left(\frac{t}{N}\right)^{\gamma/(\gamma-1)}, \tag{1}$$

$$\langle n_{\rm cov}(t)\rangle \sim B\left(\frac{t}{N}\right)^{1/(\gamma-1)}. \tag{2}$$

Here, γ is a universal exponent that depends on the lattice dimensionality and the symmetries of the interactions. We assume $\gamma>1$; the case of $\gamma=1$, realized in high dimensions, will be discussed later.

By defintion, $R(t)=1$ at $t=0$ and decreases by an amount $\delta R=1/N$ whenever a lattice site changes its value for the first time. The average rate at which sites are changed by the avalanches is $\langle n_{\rm cov}(t)\rangle/\langle S(t)\rangle$; however, only a fraction $R(t)$ of sites have not already changed from their initial state. Recalling that the active site jumps to a random part of the system between avalanches, $R(t)$ obeys the following equation:

$$\frac{dR(t)}{dt} = -\frac{\langle n_{\rm cov}(t)\rangle R(t)}{\langle S(t)\rangle N}. \tag{3}$$

Subsituting Eqs. (1) and (2) into Eq. (3) gives

$$R(t) \sim \text{const}\times t^{-\theta} \tag{4}$$

with the persistence exponent $\theta=B/A$.

Since A and B are nonuniversal constant prefactors to the algebraic growth of $\langle S(t)\rangle$ and $\langle n_{\rm cov}(t)\rangle$, respectively, one would expect that their ratio, and therefore θ, is also nonuni-

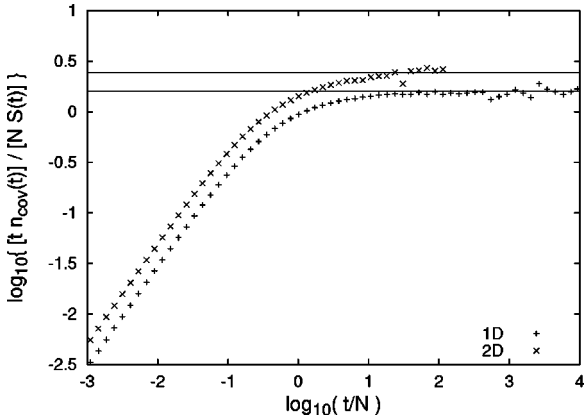

FIG. 4. The quantity $t\langle n_{\mathrm{cov}}(t)\rangle/[N\langle S(t)\rangle]$ against time for one- and two-dimensional lattices of sizes 2^{21} and $2^{11}\times 2^{10}$. The argument in the main text predicts that this will tend to a constant value that is the persistence exponent. For comparison, horizontal solid lines are plotted for $\theta=1.59$ and 2.43, which are the exponents for 1D and 2D, respectively.

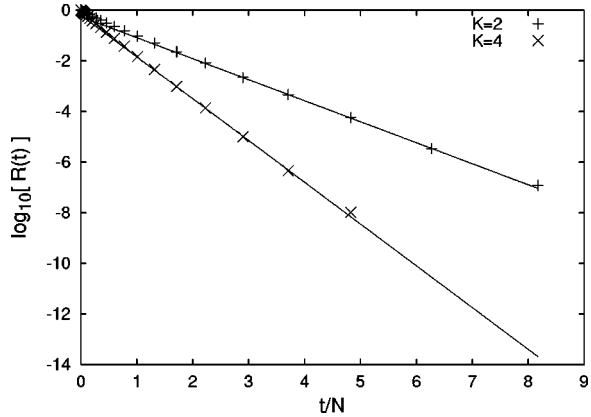

FIG. 5. $R(t)$ for a random nearest neighbor model in which the minimum and another $K-1$ sites chosen at random from the lattice are assigned new values. The system contained $N=2^{21}$ elements, and $R(t)$ was averaged over 100 runs. The solid lines have slopes of -0.83 and -1.65.

versal; however, a peculiar feature of this model is that B/A *is* universal. This can be seen by combining together two exact equations, referred to as the *gap* and *gamma* equations in [2], which gives the following equation relating $\langle S(t)\rangle$ and $\langle n_{\mathrm{cov}}(t)\rangle$:

$$\frac{d\langle S(t)\rangle}{dt}=\frac{\langle n_{\mathrm{cov}}(t)\rangle}{N}. \tag{5}$$

Substituting Eqs. (1) and (2) into Eq. (5) gives the persistence exponent as

$$\theta=\frac{B}{A}=\frac{\gamma}{\gamma-1}. \tag{6}$$

Since γ is universal, then so is θ. In one and two dimensions, $\gamma=2.70$ and 1.70, respectively [2], giving predicted values of $\theta\approx 1.59$ and 2.43, which were the values used in Figs. 1–3. As a further check on this analysis, note that Eqs. (1) and (2) predict that the quantity $t\langle n_{\mathrm{cov}}(t)\rangle/[N\langle S(t)\rangle]$ will approach the constant value $B/A=\theta$ at late times. This is confirmed by the numerical results presented in Fig. 4.

The preceding argument breaks down when $\gamma=1$, which arises for dimensions above the upper critical dimension d_{c}, for which the most recent estimate is $d_{\mathrm{c}}=4$ [21]. In this case, $\langle S(t)\rangle$ and $\langle n_{\mathrm{cov}}(t)\rangle$ both increase exponentially in time rather than algebraically [2]. By repeating the same steps as above, it is straightforward to show that $R(t)$ should now decay exponentially with a nonuniversal decay constant C,

$$R(t)\sim \mathrm{const}\times e^{-Ct}. \tag{7}$$

This is confirmed by the numerical results given in Fig. 5 for a system in which the minimum and another $K-1$ randomly chosen elements are assigned new values, which is the usual way of simulating an infinite dimensional system for this model. It is clear that the decay of $R(t)$ is exponential with a decay constant that depends strongly on K.

Finally, we consider the effects of initial conditions. All of the preceding analysis and numerical results has assumed an initial configuration in which every f_i is uniformly distributed over the range [0,1]. This can be generalized by choosing an initial state in which the f_i are uniformly distributed over the range $[f_0,1]$ and are spatially uncorrelated. It is known that the Bak-Sneppen model has a "critical value" $f_{\mathrm{c}}\in(0,1)$ such that the system is *subcritical* if $f_0<f_{\mathrm{c}}$, and *supercritical* if $f_0\geqslant f_{\mathrm{c}}$ [20]. (The value of f_{c} is nonuniversal and depends on the lattice structure and the microscopic interaction term.) We have checked that any subcritical initial state gives rise to the same persistence exponent in both one and two dimensions, as predicted by our earlier argument. However, for a supercritical initial configuration, the system enters into a single, infinite avalanche, and all of the preceding analysis fails. In fact, $R(t)$ in this situation is simply related to the growth of a single avalanche in time, which is already known to increase algebraically in time with a uni-

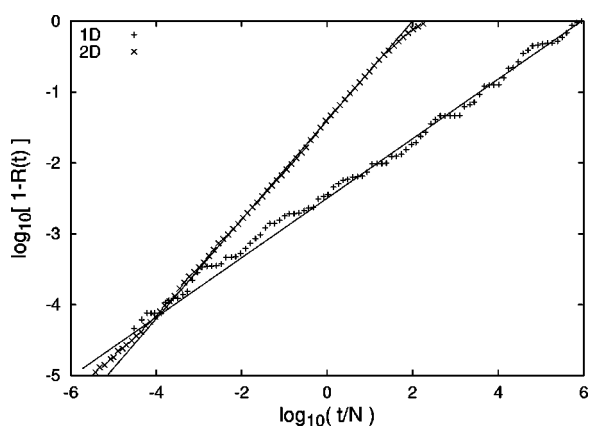

FIG. 6. $1-R(t)$ versus t/N for supercritical initial conditions in which every element has a value $f_i=1$ except for a single "seed" site that has $f_{\mathrm{seed}}<1$. The 1D data were obtained from a single run on a 2^{16} lattice, and the 2D data were averaged over 10 runs of a $2^{10}\times 2^{10}$ lattice. The solid lines give the known slopes of $\tau_{\mathrm{all}}=0.42$ in 1D and 0.70 in 2D.

versal exponent $\tau_{\rm all}$ [2]. Thus $R(t)$ does not decay algebraically but rather according to the expression

$$1-R(t)\sim \text{const}\times t^{\tau_{\rm all}} \tag{8}$$

until a late time t_0 when the avalanche ''touches'' itself through the periodic boundary conditions. Confirmation of this scaling is given in Fig. 6 for one and two dimensions. In high dimensions, $R(t)$ decays exponentially for both classes of initial state.

We have found that the persistence $R(t)$ in the Bak-Sneppen model can decay in one of three different ways. In low dimensions, it decays with a universal exponent θ when starting from a subcritical state, but by the anomalous form $1-R(t)\propto t^{\tau_{\rm all}}$ when starting from a supercritical state. For high dimensions the decay is always exponential. We also provided a simple theoretical prediction for the value of $\theta=\gamma/(\gamma-1)$, which, when combined with the numerical results, gives strong confirmation of the universality of θ.

It is not yet clear how this relates to the question of the existence of a steady state of the model, although it is interesting to note that $\theta>1$ always, which is significantly higher than typical persistence exponents, and means that the number of untouched sites will rapidly decay to zero. Thus the observed nonstationarity must be due to temporal correlations within an avalanche, rather than the spread of avalanches throughout the system.

The author would like to acknowledge the hospitality of the Vrije Universiteit, Amsterdam, where part of this work was carried out. This work was funded by EPSRC (U.K.) Grant No. GR/M09674.

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